Lap Time Optimisation for a Formula Style Car

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***Abstract*—In this paper, we explore the use of optimal control strategies to model the most efficient path for a racing car to optimize its lap time around F1 circuits, focusing on Monza (Italy) and Silverstone (United Kingdom) circuits. To achieve this, we generate a track line and add track limit constraints, to create a centre line to use as a reference. We then formulate a quadratic cost to minimize track curvature and generate the optimal race line, and use a DIRCOL-based trajectory optimization algorithm to calculate the controls required to track the trajectory. Our results demonstrate the effectiveness of the proposed optimal control strategies in achieving faster lap times, and the potential to improve racing car performance in F1 circuits. This work contributes to the broader field of optimal control and trajectory optimization in the context of high-performance motor-sports.**

I. Introduction

High-performance racing has always been an exciting and challenging field, where the goal is to achieve the fastest possible lap time around the track. In Formula One (F1) racing, drivers and engineers continuously search for ways to optimize the performance of their cars and outpace their competitors. One approach to improving lap times is to use optimal control strategies, which aim to find the most efficient path a racing car can take around the track. In this paper, we explore the use of optimal control strategies to model the most efficient path for a racing car to optimize its lap time around F1 circuits. Specifically, we focus on one of the iconic circuits: Monza in Italy.

Optimal control is a well-established field of research that seeks to find the best control inputs for a system to achieve a desired objective, subject to constraints. In F1 racing, the objective for cars is to complete a lap in the shortest possible time. However, the shortest path is not necessarily the optimal path for minimizing lap time. Similarly, the minimum curvature path, which minimizes the amount of steering required to navigate the turns, is not optimal as it does not account for the vehicle dynamics and track conditions [5]. To achieve the fastest lap times, F1 cars need to maintain maximum velocity while cornering, which requires a path that balances the trade-off between minimizing distance traveled and maximizing speed. The optimum path for an F1 car lies somewhere in between, and finding this path is a challenging problem that requires a deep understanding of optimal control, vehicle dynamics, and track geometry [6]. In this paper, we address this problem by developing a framework for using optimal control strategies to generate the optimal racing line for F1 cars on the Monza (Temple of Speed) circuit.

Our approach takes into account the vehicle dynamics and track conditions, and aims to find the path that allows the car to achieve the highest possible speed while staying within the track limits. By doing so, we aim to contribute to the broader field of high-performance motor-sports and help teams and drivers improve their lap times and race performance.

II. Literature Review

The use of minimum curvature trajectories involves finding the path with the smallest possible curvature, which minimizes the amount of time it takes to complete a turn. The approach has been extensively studied in recent years, with researchers proposing various numerical methods and optimization algorithms for computing the minimum curvature trajectory. Papers such as Siegler et al.'s 'Lap Time Simulation for Racing Car Design,' [1] Müller et al.'s "Optimisation of the Driving Line on a Race Track," [2] and Casanova, D.'s "On minimum time vehicle manoeuvring: the theoretical optimal lap" [3] provide detailed insights into the vehicle dynamics models, optimization algorithms, and numerical methods used in lap time optimization using minimum curvature trajectories. These studies demonstrate the effectiveness of the minimum curvature trajectory approach for lap time optimization and provide valuable insights into the development of lap time optimization tools using various software platforms.

Summarising our understandings, the minimum curvature trajectory approach is a popular technique for lap time optimization in motor-sports due to its numerous merits. The approach reduces the time taken to complete a turn, resulting in faster lap times. It can be seamlessly integrated with various optimization algorithms and numerical methods, providing greater flexibility in the design of the lap time optimization tool. To address the demerits of this approach, we will take a comprehensive approach in our project. We will try to take into consideration the limitations of the approach, such as the assumption that the vehicle will travel in a straight line between two turns. Overall, the minimum curvature trajectory approach is a promising technique for lap time optimization, and by taking a careful and comprehensive approach, we can effectively leverage its merits while mitigating its demerits in our project.

III. METHODOLOGY

A. Geometric Problem

The primary objective of a race driver is to achieve the fastest lap time. To accomplish this goal, there are two main strategies that can be pursued: minimizing the distance traveled or maximizing the speed attained. However, the maximum speed that can be achieved by a race car while navigating a curve with a certain radius is limited by the maximum centripetal force that can be generated by the tires. Therefore, to minimize lap time, it is essential for the driver to strike a balance between minimizing the distance traveled and maximizing the speed attained while staying within the limits of the car's centripetal force capabilities. This requires a deep understanding of the physical constraints of the vehicle, the geometry of the track, and optimal control strategies to determine the most efficient racing line. Now we give the derivation of the Quadratic problem which is inspired from [4].

The maximum speed $v_{\max}$ achievable while negotiating a curve of radius ρ is limited by the maximum centripetal force developed by the tires which can be estimated in equation (1) where m represents the vehicle's mass, μ the tire–road friction coefficient, F_{a} the aerodynamic down-force.

$$\begin{aligned} m a_{y,\max} &= m\frac{v_{\max}^2}{\rho} = \mu\left(mg + F_{\mathrm{a}}\right) \\ \Rightarrow v_{\max} &= \sqrt{\mu\rho\left(g + \frac{F_{\mathrm{a}}}{m}\right)} \end{aligned} \tag{1}$$

To further understand, please refer Fig. (1). The minimum space trajectory, labeled as (a), follows the path with the lowest curvature radius, which allows the car to cover the shortest distance. In contrast, the minimum curvature trajectory, labeled as (b), is characterized by the largest curvature radius and enables the car to negotiate the curve at the highest possible speed. However, this trajectory requires a significant increase in distance traveled. Finding the optimal racing line for minimizing lap time requires striking a balance between trajectory (a) and (b), taking into account the vehicle's dynamics.

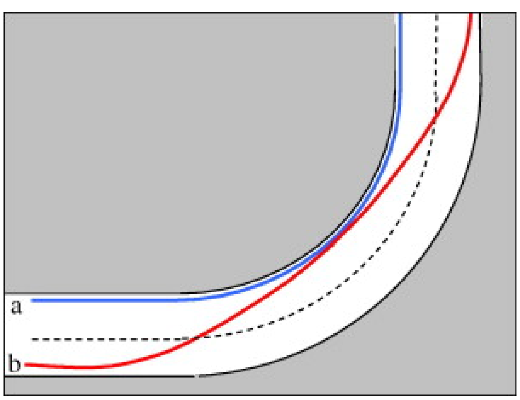

Fig. 1. Shortest path (a) and Lowest curvature (b). Taken from [4].

In this report, the approach followed involves an initial analysis of the pure geometrical problem. Algorithms are developed to identify the optimal path and trajectory with the lowest curvature based on the track center-line trajectory and road width. Subsequently, a simplified vehicle dynamics model is employed to solve a trajectory optimization problem aimed at identifying the necessary controls for the optimal trajectory.

The algorithm developed to identify minimum curvature trajectory solves a constrained minimization problem as the identified solution has to be within track limits. To solve the minimization problem, the track is divided into several segments as shown in Fig. (2) and at the end of each segment the position of a given point on the track is identified using the following equation where α_i being a parameter that identifies the position of point $\overrightarrow{\mathbf{P}}_i$ along the track width. The range of variation of α is [0:1].

$$\begin{aligned} \overrightarrow{\mathbf{P}}_i &= x_i\,\overrightarrow{\mathbf{i}} + y_i\,\overrightarrow{\mathbf{j}} \\ &= \left[x_{r,i} + \alpha_i\left(x_{l,i} - x_{r,i}\right)\right]\overrightarrow{\mathbf{i}} + \left[y_{r,i} + \alpha_i\left(y_{l,i} - y_{r,i}\right)\right]\overrightarrow{\mathbf{j}} \\ &= \left[x_{r,i} + \alpha_i \Delta x_i\right]\overrightarrow{\mathbf{i}} + \left[y_{r,i} + \alpha_i \Delta y_i\right]\overrightarrow{\mathbf{j}} \end{aligned} \tag{2}$$

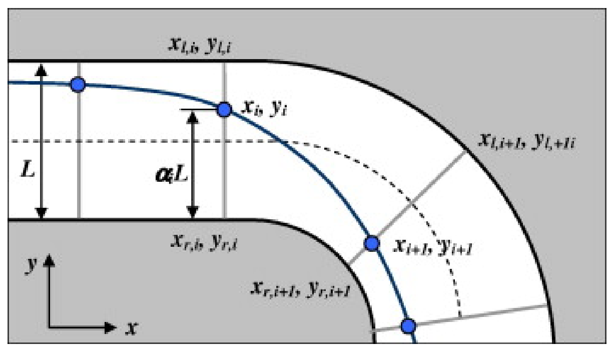

Fig. 2. Trajectory Segmentation. Taken from [4].

The resulting trajectory will be obtained by linking, through linear segments all the $\overrightarrow{\mathbf{P}}_i$ points identified.

B. Minimum Curvature Trajectory

In Formula One racing, the minimum curvature trajectory refers to the path a car takes through a turn that minimizes the distance traveled and maximizes the speed while actively maintaining vehicle control.

To follow the minimum curvature trajectory, the driver will typically approach the turn from the outside edge of the track and aim to hit the apex, or the point where the turn is at its tightest, with the inside wheels of the car. The driver will then accelerate out of the turn, using the full width of the track to maintain momentum.

Initially, the search for the minimum curvature path was approached similarly to how the shortest space trajectory is determined, which involved approximating the trajectory with a series of straight-line segments. However, this method proved to be inadequate as simulations showed significant errors in the solution due to discontinuities in slope between adjacent segments. To overcome this issue, adjacent points were instead connected using closed natural cubic splines. This approach ensured that the trajectory was determined within a single segment, thereby avoiding any curvature discontinuities that could lead to inaccuracies in the final solution.

$$\begin{cases} x_i(t) = a_{i,x} + b_{i,x}t + c_{i,x}t^2 + d_{i,x}t^3 \\ y_i(t) = a_{i,y} + b_{i,y}t + c_{i,y}t^2 + d_{i,y}t^3 \\ t(s) = \frac{s - s_{i,0}}{ds_i} \end{cases} \tag{3}$$

Equation (3) represents the trajectory within a single track segment as a third-order polynomial function of t. Here, t is a variable that represents the curvilinear abscissa normalized to the length of the i^{th} track segment and s is curvilinear distance.

$$\begin{aligned} \widehat{\Gamma}^2 &= \left(\frac{\mathrm{d}^2 x(s)}{\mathrm{d}s^2}\right)^2 + \left(\frac{\mathrm{d}^2 y(s)}{\mathrm{d}s^2}\right)^2 = \left(\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t(s)}{\mathrm{d}s}\right)^2\right)^2 \\ &+ \left(\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t(s)}{\mathrm{d}s}\right)^2\right)^2 \\ &= \left(\frac{\mathrm{d}t(s)}{\mathrm{d}s}\right)^4 \left[\left(\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2}\right)^2 + \left(\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2}\right)^2\right] \end{aligned} \tag{4}$$

If all the segments of the track center-line have the same length $ds*$, equation (4) simplifies as follows:

$$\begin{aligned} &\left(\frac{\mathrm{d}t(s)}{\mathrm{d}s}\right) = \frac{1}{\mathrm{d}s^*} \\ &\Rightarrow \quad \widehat{\Gamma}^2 = \left(\frac{1}{\mathrm{d}s^*}\right)^4 \left[\left(\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2}\right)^2 + \left(\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2}\right)^2\right] \end{aligned} \tag{5}$$

The track curvature minimization can thus be obtained by considering the quantity Γ.

$$\Gamma^2 = \sum_{i=1}^{n} \left[\left(\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2}\right)^2 + \left(\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2}\right)^2\right] \tag{6}$$

C. Formulation of Quadratic Problem

Closed natural cubic splines are a type of interpolation method used in minimum curvature problems to construct the minimum curvature trajectory. The cubic spline algorithm constructs a smooth curve by connecting several cubic polynomials together at data points. In a minimum curvature problem, the data points represent the key locations on the track, such as the center of turns and straight sections. The closed natural cubic spline approach ensures that the spline is continuous and has a continuous first and second derivative at all data points. This continuity ensures that the resulting trajectory is smooth and minimizes the overall curvature of the path, resulting in a faster lap time. The use of closed natural cubic splines has been shown to be effective in minimizing lap times and improving the performance of race cars in motor-sports applications.

The second derivative for a closed natural cubic spline, considering variable x computed at t = 0, can be expressed as

$$\left.\frac{\mathrm{d}^2 \overline{\mathbf{x}}(t)}{\mathrm{d}t^2}\right|_{t=0} = [\mathbf{D}]\overline{\mathbf{x}} \tag{7}$$

Here, [D] is a constant matrix which represents the vector of the components of each point of the trajectory. Considering equation (2), the vector $\bar{x}$ linearly depends on the independent variable vector $\bar{\alpha}$:

$$\overline{\mathbf{x}} = \overline{\mathbf{x}}_{\mathrm{r}} + [\mathbf{dx}]\overline{\boldsymbol{\alpha}} \tag{8}$$

Thus, equation (7) can be written as,

$$\begin{aligned} \left(\left.\frac{\mathrm{d}^2 \overline{\mathbf{x}}(t)}{\mathrm{d}t t^2}\right|_{t=0}\right)^2 &= \overline{\boldsymbol{\alpha}}^{\mathrm{T}} \left([\mathrm{d}\mathbf{x}]^{\mathrm{T}}[\mathbf{D}]^{\mathrm{T}}[\mathbf{D}][\mathrm{d}\mathbf{x}]\right) \overline{\boldsymbol{\alpha}} \\ &+ 2\left(\overline{\mathbf{x}}_{\mathrm{r}}^{\mathrm{T}}[\mathbf{D}]^{\mathrm{T}}[\mathbf{D}][\mathrm{d}\mathrm{x}]\right)\overline{\boldsymbol{\alpha}} + \overline{\mathbf{x}}_{\mathrm{r}}^{\mathrm{T}}[\mathbf{D}]^{\mathrm{T}}[\mathbf{D}]\overline{\mathbf{x}}_{\mathrm{r}} \end{aligned} \tag{9}$$

A similar expression can be obtained for the y coordinate so that Γ^2 can be expressed as a quadratic form of the independent variable vector $\bar{\alpha}$:

$$\Gamma^2 = \overline{\boldsymbol{\alpha}}^{\mathrm{T}} \left[\mathbf{H}_{\Gamma}\right] \overline{\boldsymbol{\alpha}} + \left\{\mathbf{B}_{\Gamma}\right\} \overline{\boldsymbol{\alpha}} + \text{ cost} \tag{10}$$

D. Trajectory Optimization as an NLP

We are now going to present the Nonlinear Program formulation of our problem to get the optimized trajectory for controls considering vehicle dynamics and track constraint. We used IPOPT NLP solver to solve the problem. The setup below gives a brief idea about the optimal control problem.

$$\min_{x_{1:N}, u_{1:N-1}} \quad \sum_{i=1}^{N-1} \left[\frac{1}{2}(x_i - x_i^{ref})^T Q (x_i - x_i^{ref}) + \frac{1}{2} u_i^T R u_i\right] \tag{11}$$

$$+ \frac{1}{2}(x_N - x_N^{ref})^T Q_f (x_N - x_N^{ref}) \tag{12}$$

$$\text{s.t.} \quad x_1 = x_{\text{IC}} \tag{13}$$

$$x_N = x_{goal} \tag{14}$$

$$f_{rk4}(x_i, x_{i+1}, u_i, dt) = 0 \quad i \in [1, N-1] \tag{15}$$

$$U_{min} \leq u_i \leq U_{max} \quad i \in [1, N-1] \tag{16}$$

$$X_{min} \leq x_i \leq X_{max} \quad i \in [1, N] \tag{17}$$

Where x_{IC} and x_{goal} is taken from the previous optimum achieved in geometric QP, and $f_{rk4}(x_i, x_{i+1}, u_i)$ is the RK4 integration of the dynamics.

1) Dynamics bicycle model: We used a single track model for the vehicle dynamics. Below are the corresponding differential equations:

TABLE I
MODEL PARAMETERS

Name	Description	Unit	Value
($\dot{x}$,$\dot{y}$)	Vehicle's velocity along vehicle's frame	m/s	State
(X,Y)	Vehicle's co-ordinates in world frame	m	State
(ψ,$\dot{\psi}$)	Body yaw angle, angular speed	rad,rad/s	State
δ	Front wheel's angle	rad	Input
F	Total input force	N	Input
M	Vehicle Mass	kg	1000
l_r	Length from rear tire to center of mass	m	0.82
l_f	Length from front tire to center of mass	m	1.18
C_α	Cornering stiffness of each tire	N	20000
I_z	Yaw inertia	$kg.m^2$	3004.5
f	Rolling resistance co-efficient	N/A	0.025
delT	Simulation time step	sec	0.01

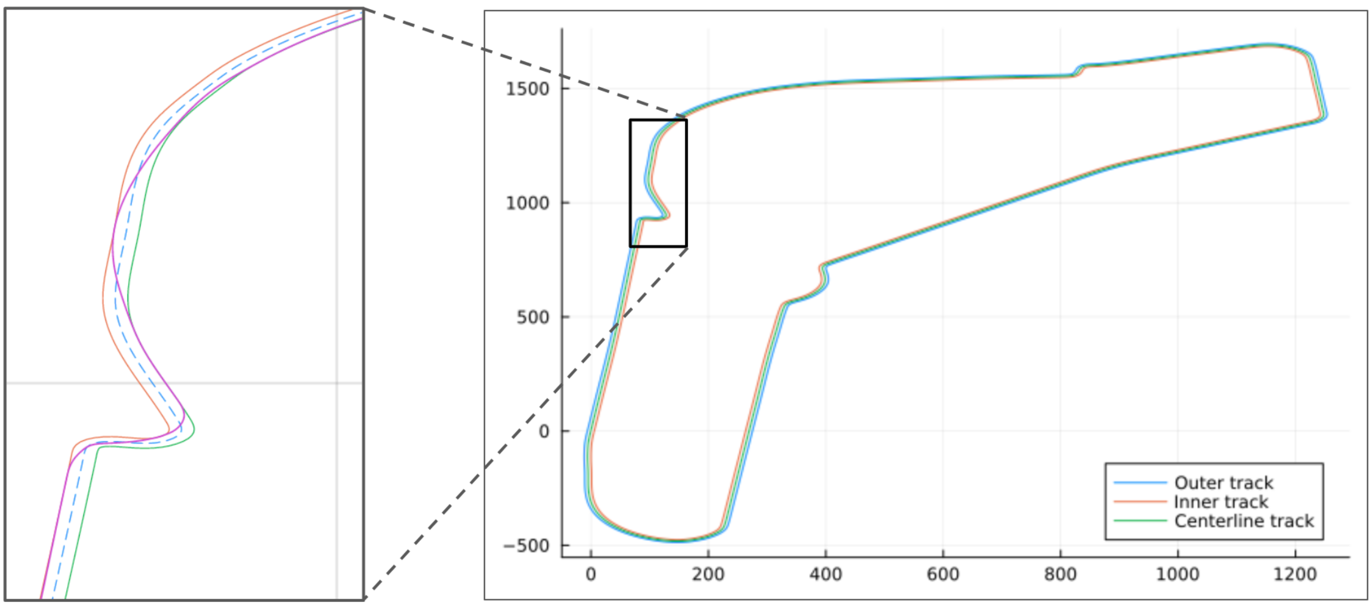

Fig. 3. Optimal Raceline using Geometric QP

$$\dot{X} = \dot{x}\cos\psi - \dot{y}\sin\psi \tag{18}$$
$$\dot{Y} = \dot{x}\sin\psi + \dot{y}\cos\psi \tag{19}$$
$$\dot{\psi} = \dot{\psi} \tag{20}$$
$$\ddot{y} = -\dot{\psi}\dot{x} + \frac{2C_\alpha}{m}\left(\cos\delta\left(\delta - \frac{\dot{y}+l_f\dot{\psi}}{\dot{x}}\right) - \frac{\dot{y}-l_r\dot{\psi}}{\dot{x}}\right) \tag{21}$$
$$\ddot{x} = \dot{\psi}\dot{y} + \frac{1}{m}(F - fmg) \tag{22}$$
$$\ddot{\psi} = \frac{2l_fC_\alpha}{I_z}\left(\delta - \frac{\dot{y}+l_f\dot{\psi}}{\dot{x}}\right) - \frac{2l_rC_\alpha}{I_z}\left(-\frac{\dot{y}-l_r\dot{\psi}}{\dot{x}}\right) \tag{23}$$

where, the state vector $X = [x, y, \psi, \dot{x}, \dot{y}, \dot{psi}]$ and the controls U are F (Force) and δ (steering angle). $U_{min} = [-20000, -\pi/6]$ and $U_{max} = [16000, \pi/6]$. X_{min} and X_{max} is track limit constraint . The model parameters are explained in the Table: I

IV. RESULTS

The objective of this study was to identify the most efficient trajectory for a given track, using minimum curvature optimization techniques. This section presents the data collected during the experimentation, as well as the results of the analysis performed to validate our findings.

A. Quadratic Problem results

The results for QP problem are shown in Fig. (3) and Fig. (4). Curvature plot shows us zero curvature in steps corresponding to straights in the track. An inference drawn from the results of the Quadratic problem is that traditional optimization techniques, such as quadratic programming, may not always be suitable for trajectory optimization in motorsports applications due to limitations such as discretization errors. The study found that the discretization of the trajectory resulted in the path touching the apex, which can lead to performance issues during actual racing scenarios. These findings suggest that more advanced optimization techniques and a nuanced understanding of the complex factors that affect trajectory optimization in motor-sports are necessary to improve lap times and enhance the performance of race cars. Overall, this study highlights the need for continued research into trajectory optimization techniques and their applications in motor-sports.

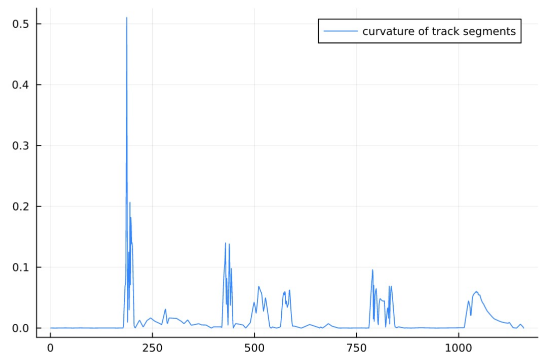

Fig. 4. Curvature Plot of Raceline obtained from Geometric QP

B. Trajectory Optimization NLP

The results derived from DIRCOL suggest that the use of direct collocation techniques can effectively optimize the trajectory of a Formula-style car. The DIRCOL algorithm was able to develop a smooth and continuous trajectory that minimized lap times while satisfying the constraints of the problem. The optimized trajectory also resulted in consistent velocity and acceleration profiles, which could enhance the performance of the car on the track. These findings suggest that direct collocation techniques have potential for improving lap times and enhancing the overall performance of race cars. However, further research is necessary to explore the full capabilities and limitations of these techniques in the context of motor-sports applications.

Fig. (5) in the present study demonstrates the discrepancy in distance between the trajectory generated by Quadratic Programming (QP) and the trajectory solved by Nonlinear Programming (NLP), which were found to be fairly close. However, it is noteworthy that the NLP method accounted for both the dynamics of the car and the conditions of the track, indicating its superiority over the QP approach in optimizing the trajectory of Formula-style cars.

V. Conclusion

In conclusion, the results of this project demonstrate the effectiveness of closed natural cubic splines for developing a minimum curvature trajectory that improves lap times and enhances the performance of a Formula-style car. The optimized trajectory significantly reduces the overall curvature of the path, resulting in a smoother ride and faster lap times. This study has important implications for motor-sports applications and suggests that minimum curvature optimization techniques can be used to improve lap times and performance in Formula-style car racing.

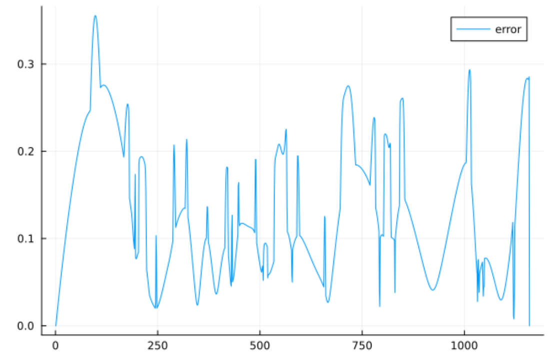

Fig. 5. Distance Error between Qp and NLP solved trajectory

However, the study has some limitations, such as the need for accurate data collection and the potential for local minima in the optimization process. Future research could address these limitations by exploring the use of other optimization techniques, such as genetic algorithms, to develop more accurate and efficient minimum curvature trajectories. Additionally, future studies could examine the effects of other variables, such as tire pressure, on lap times and performance, to further refine the trajectory optimization process. Overall, the results of this project provide a valuable foundation for future research in the field of motors-ports and trajectory optimization.

VI. Appendix

Project Repository : Repo Link

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